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String hamiltonian from generalized YM gauge theory in two dimensions

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ABSTRACT: Two dimensional $SU(N)$ Yang-Mills theory is known to be equivalent to a string theory, as found by Gross in the large N limit, using the $1/N$ expansion. Later it was found that even a generalized YM theory leads to a string theory of the Gross type. In the standard YM theory case, Douglas and others found the string hamiltonian describing the propagation and the interactions of states made of strings winding on a cylindrical space-time. We address the problem of finding a similar hamiltonian for the generalized YM theory. As in the standard case we start by writing the theory as a theory of free fermions. Performing a bosonization, we express the hamiltonian in terms of the modes of a bosonic field, that are interpreted as in the standard case as creation and destruction operators for states of strings winding around the cylindrical space-time. The result is similar to the standard hamiltonian, but with new kinds of interaction vertices.

KEYWORDS: String-YM correspondence, two-dimensional Yang-Mills.

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1. Introduction

The idea of a relation between gauge theories and string theories is very old, and a lot of progresses were made in this direction in the last years, with the introduction of the Maldacena conjecture and the subsequent work.

A less recent approach to the problem is to study a simpler model, the pure two dimensional $SU(N)$ Yang-Mills theory, which is exactly solvable and can be described as a string theory [1]-[9]. In light of the recent developments, it is interesting to reconsider this model, because it can be useful to gain a better insight on the string/YM correspondence.

Two-dimensional $SU(N)$ Yang-Mills theory does not contain gluons, nevertheless it is not trivial but there are degrees of freedom related to the topology of the manifold [10]. The interesting feature of this theory is that the partition function can be exactly calculated on an arbitrary two dimensional manifold [11, 12].

The interpretation in terms of a string theory can be achieved in two ways:

- the partition function of the theory on a two dimensional manifold Σ_T for large N can be developed in powers of $1/N$ and in this way it can be interpreted as the partition function of a string theory with two-dimensional space-time Σ_T , coupling constant $g_s = 1/N$ and string tension $\lambda = g^2 N$ (where g is the coupling constant of the gauge theory) [5, 6, 7];
- the two-dimensional Yang-Mills theory is equivalent to a system of free non-relativistic fermions that in turn, for large N , can be rewritten as relativistic ones; bosonizing one obtains a string description [1, 3, 4]: one can thus calculate the string hamiltonian starting from the Yang-Mills one.

The starting point of our investigation is the fact that in two dimensions the Yang-Mills action can be generalized to a class of theories; this is not possible in four dimensions where the generalized theories are non-renormalizable. Since YM_2 is just a special point in this space of theories, it is interesting to investigate if also the other theories can be described in terms of strings.

That this is indeed the case has been seen performing a $1/N$ expansion of the partition function analogous to that of Gross and Taylor [13, 14]. However the computation of the string hamiltonian by bosonizing the fermionic description of the theory is a much more straightforward way to obtain the string description; in this paper we present the result we have obtained following this approach.

The paper is organized as follows: In section 2 we introduce the generalized theory, underlining the fact that the generalized hamiltonian in the representation basis is simply a combination of higher Casimirs; then we study its equivalence with a fermionic system, closely following the standard case analysis and introducing the formulas that we have derived for the calculation of the higher Casimirs. In the following sections we focus on the quartic Casimir case, since we think that this example is sufficient to understand the general features of the generalized theories and to draw some conclusions. In particular in section 3 we express the quartic Casimir hamiltonian in terms of non-relativistic fermionic fields, then in terms of relativistic fermions and finally in terms of bosonic field modes, thus obtaining the string hamiltonian. Then we discuss the result and we make some considerations on the general case.

2. YM_2 generalized and description in terms of free fermions

To introduce the generalization of the Yang-Mills theory in two dimensions, we need first to recall the definition of the standard theory. The YM_2 action is

$$\mathcal{I} = -\frac{1}{2g^2} \int d^2x \sum_a F_a^{\mu\nu} F_{\mu\nu a} , \quad (2.1)$$

where a labels the generators of the group of internal symmetries. In the following we consider first $G = U(N)$, then we extend the results for the Yang-Mills theory for $SU(N)$.¹

We consider the quantization on the cylinder with periodic spatial coordinate of period L . As we work in two dimensions, $F_a^{\mu\nu}$ has a single degree of freedom $F_a^{10} = E_a$; in the canonical way $E_a = -ig^2 \frac{\delta}{\delta A_a^1}$. Fixing the gauge by $A_0 = 0$ the equation of motion gives the constraint $D_1 F_{10} \Psi = 0$ that tells us that the wave-function Ψ is simply a class function in the holonomy variable $U = P \exp(i \int_0^L A_1 dx)$. The natural scalar product between wave-functions is

$$(\Psi_1, \Psi_2) = \int_G dU \Psi_1^*(U) \Psi_2(U) , \quad (2.2)$$

where dU is the invariant (Haar) measure on gauge group G .

¹We calculate the corresponding string hamiltonian only for the $SU(N)$ theory, since in this case (as for the standard theory [15]) the result and the string interpretation are simpler.

With this scalar product, a natural basis in the Hilbert space is the irreducible representations basis $\{|R\rangle\}$, with the corresponding wave-functions:

$$\langle U|R\rangle = \chi_R(U) = \text{Tr}_R(U) . \quad (2.3)$$

The hamiltonian takes the heat kernel form (see [16]):

$$H = \int_0^L dx \frac{1}{2g^2} \sum_a E_a E_a = - \int_0^L \frac{g^2}{2} \sum_a \frac{\delta}{\delta A_a^1} \frac{\delta}{\delta A_a^1} . \quad (2.4)$$

Using $\frac{\delta}{\delta A_a^1} \chi_R(U) = i \chi_R(T_a U)$ (where $\{T_a\}$ is a set of generators of the group) and the expression for the quadratic Casimir $C_2 = \sum_a T_a T_a$, in the basis (2.3) the hamiltonian becomes:

$$H = \frac{g^2}{2} L C_2(R) = \frac{\lambda L}{2N} C_2(R) , \quad (2.5)$$

where $\lambda = Ng^2$ (the t'Hooft coupling) is kept fixed in the $N \rightarrow \infty$ limit.

The generalization of the theory is based on the fact that the partition function of the standard theory

$$Z = \int [DA^\mu] e^{-\frac{1}{2g^2} \int d^2x \sum_a F_a^{\mu\nu} F_{\mu\nu a}} \quad (2.6)$$

is equivalent to the one built from the action

$$\mathcal{I}' = -\frac{1}{4} \int d^2x \left(i \phi^a \epsilon^{\mu\nu} F_{\mu\nu a} + \frac{g^2}{2} \sum_a \phi_a \phi_a \right) , \quad (2.7)$$

where ϕ is an auxiliary scalar field [10]. This formulation gives the possibility to generalize the YM₂ theory replacing in \mathcal{I}' the term $\frac{g^2}{2} \sum_a \phi_a \phi_a$ with terms of higher degrees in ϕ [10]. Eliminating the auxiliary field from the partition function one can work out hamiltonians similar to (2.4) (but with more than two derivatives). With such hamiltonians one can build a generalized heat kernel equation [13]. Writing this equation in terms of the holonomy variable U , one finally obtains the new hamiltonian looking like (2.5) but with a linear combination of higher Casimirs instead of the quadratic one:

$$H = \sum_k \frac{\lambda_k L}{N^{k-1}} C_k , \quad (2.8)$$

where $\lambda_k = g_k^2 N^{k-1}$ is the coupling to be held fixed in the large N limit (see [13, 14]).

It is important to notice that this procedure gives only symmetrized Casimirs (in the sense specified in Sect. 2.2), see [13].

2.1 Free fermions for a generic Casimir invariant

To study the equivalence of the generalized YM theory with a system of non-relativistic fermions, we first briefly summarize the method used in the standard case [17, 3].

Since class functions are determined by their values on the maximal torus ($U = \text{Diag}(e^{i\theta_1}, \dots, e^{i\theta_N})$) and they are invariant for exchange of the eigenvalues, they are totally symmetric functions $\Psi(\theta_1, \dots, \theta_N)$. The scalar product becomes:

$$(\Psi_1, \Psi_2) = \frac{1}{N!(2\pi)^N} \int \prod d\theta_i \tilde{\Delta}(\vec{\theta})^2 \Psi_1^*(\vec{\theta}) \Psi_2(\vec{\theta}) , \quad (2.9)$$

where $\tilde{\Delta}(\vec{\theta}) = \prod_{i < j} \sin \frac{\theta_i - \theta_j}{2}$. In the standard case the hamiltonian in terms of the $\vec{\theta}$ is

$$H = \frac{\lambda}{2N} L \hat{C}_2 = \frac{\lambda}{2N} L \frac{1}{\tilde{\Delta}} \left[\sum_i \left(-\frac{\partial^2}{\partial \theta_i^2} \right) - F_2 \right] \tilde{\Delta}, \quad (2.10)$$

with $F_2 = \frac{N(N^2-1)}{12}$.

Looking at the hamiltonian and the scalar product, one can see that it is better to use as wave-functions the anti-symmetric $\psi(\vec{\theta}) = \tilde{\Delta}(\vec{\theta}) \Psi(\vec{\theta})$, obtaining in this way a non-relativistic theory of N free fermions on the circle.

The generic base state $|R\rangle$ with associated Young diagram with row lengths h_i corresponds in this fermionic theory to the base state $\psi_{\vec{n}}(\vec{\theta}) = \det_{1 \leq i, j \leq N} e^{i n_j \theta_i}$ with $n_j = h_j - j + n_F + 1$ (where $n_F = \frac{N-1}{2}$ is the Fermi level).

From the eigenvalues equation for \hat{C}_2 :

$$\hat{C}_2 \Psi_R = C_2(R) \Psi_R, \quad (2.11)$$

we obtain the equation for ψ :

$$C_2(R) \psi_R = C_2(R) \tilde{\Delta} \Psi_R = \tilde{\Delta} \hat{C}_2 \Psi_R = -\partial_i^2 (\tilde{\Delta} \Psi_R) - F_2 \tilde{\Delta} \Psi_R = -\partial_i^2 \psi_R - F_2 \psi_R \quad (2.12)$$

that is

$$\hat{C}_2' \psi_R \equiv -\partial_i^2 \psi_R - F_2 \psi_R = \left[\sum_i n_i^2 - F_2 \right] \psi_R = C_2(R) \psi_R. \quad (2.13)$$

Similarly for the higher Casimirs the same wave-functions Ψ satisfy

$$\hat{C}_k \Psi_R = C_k(R) \Psi_R, \quad (2.14)$$

that in terms of $\psi = \tilde{\Delta} \Psi$ becomes

$$\hat{C}_k' \psi_R = \tilde{\Delta} \hat{C}_k \left(\frac{1}{\tilde{\Delta}} \psi_R \right) = C_k(R) \psi_R, \quad (2.15)$$

i.e. an equation analogous to (2.13).

We are not interested in writing the \hat{C}_k operator, because we can start directly from the eigenvalues of the fermionic hamiltonian $C_k(R)$, which are simply the Casimir operators in terms of the n_i 's. In particular we focus on the theory with a quartic Casimir, since it is the simplest generalization, sufficient to show the new features of the general case. A cubic Casimir or a Casimir of order $k = 2m + 1$ (odd) is not suitable since it has a leading $n_i^k = n_i^{(2m+1)}$ term for $n_i \rightarrow -\infty$, thus the eigenvalues of the hamiltonian are not bounded from below, and therefore the theory is not quantizable.

2.2 Calculation of the higher Casimirs

In this subsection we show how to write $C_k(R)$ as a function of the $\{n_i\}$, and this is the starting point for the subsequent analysis as stated above. To obtain this expression for $C_k(R)$, we start from the higher Casimir operators illustrated in [1], Sect. 4.10.

The k -th symmetrized Casimir operator for a representation of rank n of $U(N)$ (that is built from the direct product of n copies of the fundamental representation) is

$$\hat{C}_k = \sum_{i_1 \dots i_k} \sum_{a_1 \dots a_k} E_{i_1 i_2}^{(a_1)} E_{i_2 i_3}^{a_2} \dots E_{i_k i_1}^{a_k}, \quad (2.16)$$

where the i 's are in $1, 2, \dots, N$, the a 's label the n spaces and E_{ij}^a is the (i, j) -generator in the Lie algebra of the a -th space. (\dots) means total symmetrization.²

The generators E_{ij}^a are given by

$$(E_{ij}^a)_{\alpha\beta} = \delta_{i\alpha} \delta_{j\beta}, \quad (2.17)$$

and obey the following commutation rules

$$[E_{ij}^a, E_{kl}^b] = 0 \quad \text{if } a \neq b \quad [E_{ij}^a, E_{kl}^a] = \delta_{jk} E_{il}^a - \delta_{il} E_{kj}^a. \quad (2.18)$$

We need the eigenvalue of the Casimir operator for a particular representation R of rank n . This can be found as $C_k(R) = \chi_R(\hat{C}_k)/d_R$.

The first Casimir is

$$C_1 = \sum_i \sum_a E_{ii}^a = n. \quad (2.19)$$

To write higher Casimirs, we use (see [1]):

$$\sum_{i_1 \dots i_k} \sum_{\substack{a_1 \dots a_k \\ \forall i \neq j, a_i \neq a_j}} E_{i_1 i_2}^{a_1} E_{i_2 i_3}^{a_2} \dots E_{i_k i_1}^{a_k} = P_k, \quad (2.20)$$

where $P_k = kT_k$, and T_k is the sum of the elements in the class of permutations in S_n with a cycle of length k and the remainings of length 1.

When we evaluate this on a representation we obtain

$$\gamma_k \equiv \frac{\chi_R(P_k)}{d_R} = k \frac{\chi_R(T_k)}{d_R}. \quad (2.21)$$

Expanding the Casimir operators in terms of the P_k , it turns out that they can be expressed as functions of C_1 and some γ_k 's. For example for the quadratic Casimir (where the symmetrization is not needed because we are tracing, so the operator is automatically symmetrized) we reobtain the formula of the previous section:

$$\begin{aligned} \hat{C}_2 &= \sum_{i_1 i_2} \sum_{a_1 a_2} E_{i_1 i_2}^{a_1} E_{i_2 i_1}^{a_2} = \\ &= \sum_{i_1 i_2} \sum_{a_1 \neq a_2} E_{i_1 i_2}^{a_1} E_{i_2 i_1}^{a_2} + \sum_{i_1 i_2} \sum_a E_{i_1 i_2}^a E_{i_2 i_1}^a \Rightarrow \\ C_2 &= \gamma_2 + \sum_{i_1 i_2} \sum_a \delta_{i_2 i_1} E_{i_1 i_1}^a = \gamma_2 + N C_1. \end{aligned} \quad (2.22)$$

²Non-symmetrized Casimirs on one hand are excluded from YM_2 considerations as stated above, and on the other hand they give a string hamiltonian asymmetric under the exchange $\alpha_n \leftrightarrow \tilde{\alpha}_n$, thus a hamiltonian that unnaturally distinguishes between a winding direction and the opposite. This is again an hint of correspondence between the Yang-Mills and the string theories.

Isolating the various cases of equal and unequal indexes yields the desired form for the Casimirs, expressed with γ_2, γ_3 etc.. For example we have for the 3rd non-symmetric Casimir:

$$C_3^{(non-sym)} = \gamma_3 + 2N\gamma_2 + C_1(C_1 - 1) + N^2 C_1. \quad (2.23)$$

And the 3rd symmetric Casimir:

$$C_3 = \gamma_3 + \frac{3N}{2}\gamma_2 + \frac{3}{2}C_1(C_1 - 1) + \frac{N^2 + 1}{2}C_1 = \gamma_3 + \frac{3N}{2}\gamma_2 + \frac{3}{2}C_1^2 + \frac{N^2 - 2}{2}C_1. \quad (2.24)$$

A very long calculation gives the 4th symmetric Casimir:

$$C_4 = \gamma_4 + 2N\gamma_3 + \left(4C_1 + \frac{7N^2}{6} - 5\right)\gamma_2 + \frac{17N}{6}C_1^2 + \frac{N(N^2 - 12)}{6}C_1. \quad (2.25)$$

At this point we need the expression of these invariants in terms of the data that specify a single representation, i.e. the free fermions momenta $\{n_i\}$ that are related to the row lengths $\{h_i\}$ of the Young diagram by $\{n_i = h_i + \frac{N+1}{2} - i\}$. Same intermediate calculations are made with the set $\{l_i = h_i + N - i = n_i + \frac{N-1}{2}\}$.

The $\{\gamma_k\}$ are expressed in terms of the $\{l_i\}$ (see the App. A, where a general method to calculate a γ_k with a given k is explained):

$$\begin{aligned} \gamma_1 &= \sum_i l_i - \binom{N}{2} \\ \gamma_2 &= \sum_i l_i^2 - (2N - 1) \sum_i l_i + \frac{N(N - 1)(2N - 1)}{3} \\ \gamma_3 &= \sum_i l_i^3 - 3N \sum_i l_i^2 - 3 \sum_{i < j} l_i l_j + \frac{9N^2 - 9N + 4}{2} \sum_i l_i + \\ &\quad - \frac{3}{8}N(N - 1)(3N^2 - 3N + 2) \\ \gamma_4 &= \sum_i l_i^4 - 4 \sum_{i < j} (l_i^2 l_j + l_i l_j^2) - (4N + 2) \sum_i l_i^3 + (16N - 8) \sum_{i < j} l_i l_j + \\ &\quad + (8N^2 + 3) \sum_i l_i^2 - \frac{2}{3}(2N - 1)(8N^2 - 8N + 9) \sum_i l_i + \\ &\quad + \frac{4}{7}N(N - 1)(2N - 1)(4N^2 - 4N + 7). \end{aligned} \quad (2.26)$$

With this expressions we arrive to the expressions for the Casimir invariants in terms of $\{n_i\}$. The quadratic Casimir is

$$C_2^{U(N)} = NC_1 + \gamma_2 = \sum_i l_i^2 - (N - 1) \sum_i l_i + \binom{N}{3} = \sum_i n_i^2 - F_2, \quad (2.27)$$

while the quartic is

$$C_4^{U(N)} = \sum_i l_i^4 - 2(N + 1) \sum_i l_i^3 - 4 \sum_{i < j} (l_i^2 l_j + l_i l_j^2) +$$

$$\begin{aligned}
& + \left(\frac{7N^2}{6} + \frac{29N}{6} - 2 + 4 \sum_j l_j \right) \sum_i l_i^2 - 4(2N-1) \left(\sum_i l_i \right)^2 + \\
& + \left(\frac{47N}{3} - 8 \right) \sum_{i < j} l_i l_j - \binom{N}{2} \frac{17N^3 + 17N^2 - 108N + 72}{180} \\
& = \sum_i n_i^4 + \frac{3-2N^2}{6} \sum_i n_i^2 - \frac{N}{6} \left(\sum_i n_i \right)^2 + \frac{N(N^2-1)(11N^2-9)}{720}.
\end{aligned} \tag{2.28}$$

Casimirs of $SU(N)$ come from the shift (notice that this assures translation invariance of the $SU(N)$ Casimirs)³:

$$n_i \rightarrow n_i - \frac{\sum_j n_j}{N}, \tag{2.29}$$

From this substitution we have the 4th symmetric Casimir of $SU(N)$:

$$\begin{aligned}
C_4^{SU(N)} &= \sum_i n_i^4 - \frac{4(\sum_i n_i)(\sum_j n_j^3)}{N} + \frac{2N^2-3}{6N} \left(\sum_i n_i \right)^2 + \\
& + \left(\frac{6(\sum_j n_j)^2}{N^2} - \frac{2N^2-3}{6} \right) \sum_i n_i^2 - \frac{3(\sum_i n_i)^3}{N^3} + \frac{N(N^2-1)(11N^2-9)}{720}.
\end{aligned} \tag{2.30}$$

More generally this procedure could produce any desired symmetric or non-symmetric higher Casimir invariant for $U(N)$ (or $SU(N)$). We will see that the quartic Casimir is sufficient to show many peculiarities of the higher Casimir case.

3. String hamiltonian from the generalized YM_2 with Quartic Casimir

In this section we repeat the steps that lead to the string hamiltonian in the standard case, starting from the generalized Yang-Mills theory with a quartic Casimir. The notations and the methods are those of [15].

As in the case of quadratic Casimir, we write the YM theory as a system of free non-relativistic fermions, then in the large N limit the fermionic theory is written as a relativistic theory of fermions, a bc theory that can be bosonized. As in the standard case the bosonic creation and destruction operators are interpreted as the operators that create and destroy states of strings with a given winding on the cylinder.

First of all we introduce the second quantization formalism for the non-relativistic fermions, by means of the operators B_n and B_n^\dagger that destroy and create a fermion in the n -th level, with the anticommutation relations:

$$\{B_n, B_m^\dagger\} = \delta_{n,m}, \tag{3.1}$$

and the constraint that the number of particle is N :

$$\sum_n B_n^\dagger B_n = N. \tag{3.2}$$

³For a more precise derivation see [1], Sect. 4.10

The fundamental state of the theory must satisfy the relations:

$$\begin{aligned} B_n |0\rangle_F &= 0 \text{ if } |n| > n_F \\ B_n^\dagger |0\rangle_F &= 0 \text{ if } |n| \leq n_F, \end{aligned} \quad (3.3)$$

and the normal ordering $::$ is defined with respect to this vacuum state.

With this normal ordering, the constraint (3.2) becomes:

$$\sum_n : B_n^\dagger B_n : = 0. \quad (3.4)$$

Using the prescription

$$\sum_i n_i^k \rightarrow \sum_n n^k B_n^\dagger B_n = \sum_n n^k : B_n^\dagger B_n : + F_k, \quad (3.5)$$

with⁴ $F_k = \sum_{n=-n_F}^{+n_F} n^k$, we can write the quartic Casimir (2.30) as (we see that the pure constant terms cancel, as they should since $H|0\rangle_F = 0$):

$$\begin{aligned} C_4 &= \sum_n n^4 : B_n^\dagger B_n : - \frac{4}{N} \sum_n n : B_n^\dagger B_n : \sum_{n'} n'^3 : B_{n'}^\dagger B_{n'} : + \\ &+ \frac{6}{N} \left(\sum_n n : B_n^\dagger B_n : \right)^2 \sum_{n'} n'^2 : B_{n'}^\dagger B_{n'} : + \frac{N^2 - 1}{2} \left(\sum_n n : B_n^\dagger B_n : \right)^2 + \\ &- \frac{3}{N^3} \left(\sum_n n : B_n^\dagger B_n : \right)^4 + N \sum_n n^3 : B_n^\dagger B_n : + \\ &- 3 \left(\sum_n n : B_n^\dagger B_n : \right) \left(\sum_{n'} n'^2 : B_{n'}^\dagger B_{n'} : \right) + \\ &- \frac{N(N^2 - 1)}{4} \sum_n n : B_n^\dagger B_n : + \frac{2}{N} \left(\sum_n n : B_n^\dagger B_n : \right)^3 + \\ &+ \frac{1}{2} \sum_n n^2 : B_n^\dagger B_n : - \frac{1}{2N} \left(\sum_n n : B_n^\dagger B_n : \right)^2. \end{aligned} \quad (3.6)$$

3.1 Relativistic fermions, bc theory

As in [15], at large N , we can express the fermionic operators B_n and B_n^\dagger in terms of the bc and $\tilde{b}\tilde{c}$ fields as

$$\begin{aligned} b_n &= B_{n_F+1+n}, & c_n &= B_{n_F+1-n}^\dagger, \\ \tilde{b}_n &= B_{-n_F-1-n}, & \tilde{c}_n &= B_{-n_F-1+n}^\dagger. \end{aligned} \quad (3.7)$$

In the large N limit the cut-off on the modes of b, c, \tilde{b} and \tilde{c} , can be removed, as the missing contributions are non-perturbative (see [15] for a discussion of this point); hence here $n \in \{-\infty, +\infty\}$ ⁵.

⁴We have $F_{2l+1} \equiv 0$, F_2 as above is $\frac{N(N^2-1)}{12}$ while $F_4 = \frac{N(N^2-1)(3N^2-7)}{240}$.

⁵Thus as in [15] we can use the usual bosonization formulas. Here we do not set to zero the modes b_n, c_n with $|n| > n_F$, only for simplicity in the notation.

The $bc, \tilde{b}\tilde{c}$ fields satisfy the anticommutation relations:

$$\begin{aligned} \{c_n, b_m\} &= \{\tilde{c}_n, \tilde{b}_m\} = \delta_{n+m,0} \\ \{c_n, \tilde{b}_m\} &= \{\tilde{c}_n, b_m\} = 0 \end{aligned} \quad (3.8)$$

and the constraint:

$$\sum_n (: c_{-n} b_n : + : \tilde{c}_{-n} \tilde{b}_n :) = 0. \quad (3.9)$$

From these operators we can build the fields:

$$b(z) = \sum_n \frac{b_n}{z^{n+\lambda}}, \quad c(z) = \sum_n \frac{c_n}{z^{n+1-\lambda}}, \quad (3.10)$$

$$\tilde{b}(\bar{z}) = \sum_n \frac{\tilde{b}_n}{\bar{z}^{n+\lambda}}, \quad \tilde{c}(\bar{z}) = \sum_n \frac{\tilde{c}_n}{\bar{z}^{n+1-\lambda}}, \quad (3.11)$$

where z is a complex variable, while λ is a real parameter: we consider it generic since it is a good check to verify that the hamiltonian does not depend on it (as it should, since it is an auxiliary parameter). From the anticommutation relations we can see that this is a bc and $\tilde{b}\tilde{c}$ theory (see [18], sect. 2.7); moreover the vacuum $|0\rangle_F$ is the ground state $|\downarrow\rangle$.

We have to write expressions like (3.5) in terms of the b, c modes:

$$\sum_n n^k : B_n^\dagger B_n := \sum_n (n_F + 1 + n)^k \left[: c_{-n} b_n : + (-1)^k : \tilde{c}_{-n} \tilde{b}_n : \right]. \quad (3.12)$$

Using

$$(n_F + 1 + n)^k = \sum_{i=0}^k n^i \left(\frac{N+1}{2} \right)^{k-i} \binom{k}{i}, \quad (3.13)$$

we have to calculate sums like

$$\sum_n n^i : c_{-n} b_n : \quad (3.14)$$

(and similar for tilded fields). The (3.14) can be expressed easily as a linear combination of integrals of type:

$$W_{ab} \equiv \oint \frac{dz}{2\pi i} z^{a+b} : \partial^a c(z) \partial^b b(z) :, \quad (3.15)$$

since their expression in terms of the modes is

$$W_{ab} = (-1)^b \sum_n \prod_{i=-a}^{b-1} (n + \lambda + i) : c_{-n} b_n :. \quad (3.16)$$

Inverting (3.16) yields

$$\begin{aligned} \sum_n n : c_{-n} b_n : &= -W_{01} - \lambda W_{00} \\ \sum_n n^2 : c_{-n} b_n : &= W_{11} + (2\lambda - 1)W_{01} + \lambda^2 W_{00} \end{aligned}$$

$$\begin{aligned}
\sum_n n^3 : c_{-n} b_n : &= W_{12} + 3\lambda W_{11} - (3\lambda^2 - 3\lambda + 1)W_{01} - \lambda^3 W_{00} \\
\sum_n n^4 : c_{-n} b_n : &= W_{22} - 2(2\lambda - 1)W_{12} - (6\lambda^2 + 1)W_{11} + \\
&\quad + (4\lambda^3 - 6\lambda^2 + 4\lambda - 1)W_{01} + \lambda^4 W_{00}.
\end{aligned} \tag{3.17}$$

The antichiral expression are the same, all with tilded objects.

At this point we have the expressions of $\sum_n n^k : B_n^\dagger B_n :$ for $n = \{1, 2, 3, 4\}$, as functions of $W_{00}, W_{01}, W_{11}, W_{12}$ and W_{22} .

3.2 Bosonization

The last step is to bosonize the bc and $\widetilde{b}\widetilde{c}$ theory in the standard way (see [18], sect. 10.7). First we must notice that to bosonize we need the conformal ordering instead of the annihilation-creation one; in the bc theory there is the relation:

$$: b(z)c(z') :_c = : b(z)c(z') : + \frac{z^{1-\lambda} z'^{\lambda-1} - 1}{z - z'} \tag{3.18}$$

(and similar for $\widetilde{b}\widetilde{c}$).

We can now bosonize in terms of the field $X(z, \bar{z}) = X_L(z) + X_R(\bar{z})$:

$$\begin{aligned}
b(z) &= : e^{iX_L(z)} :_c \\
c(z) &= : e^{-iX_L(z)} :_c \\
\widetilde{b}(\bar{z}) &= : e^{iX_R(\bar{z})} :_c \\
\widetilde{c}(\bar{z}) &= : e^{-iX_R(\bar{z})} :_c.
\end{aligned} \tag{3.19}$$

For $\lambda = 1/2$ we obtain a scalar free field theory; for λ generic this is a linear dilaton CFT.

We can expand the bosonic field in modes:

$$\begin{aligned}
\partial X_L(z) &= i \sum_{n \in \mathbb{Z}} \frac{\alpha_n}{z^{n+1}} \\
\bar{\partial} X_R(\bar{z}) &= i \sum_{n \in \mathbb{Z}} \frac{\widetilde{\alpha}_n}{\bar{z}^{n+1}},
\end{aligned} \tag{3.20}$$

which satisfy the commutation relations:

$$\begin{aligned}
[\alpha_m, \alpha_n] &= [\widetilde{\alpha}_m, \widetilde{\alpha}_n] = m\delta_{m+n,0} \\
[\alpha_m, \widetilde{\alpha}_n] &= 0.
\end{aligned} \tag{3.21}$$

Following the Gross-Taylor interpretation of the standard Yang-Mills partition function in terms of a string theory [5, 6, 7] and the subsequent construction of the hamiltonian by Minahan and Polychronakos [8], the modes α_n with $n > 0$ ($n < 0$) can be interpreted as operators of annihilation (creation) of a string winding n times around the spatial circle in the same sense as the orientation; similarly the modes $\widetilde{\alpha}_n$ describe strings winding in the opposite sense. Notice that this interpretation does not depend on λ as it should be; we will check that the same happens for our hamiltonian from the quartic Casimir.

Finally the constraint (3.9) in terms of the bosonic modes is:

$$\alpha_0 + \tilde{\alpha}_0 = 2(\lambda - 1). \quad (3.22)$$

To bosonize, we need first to express W_{ab} in terms of

$$W_{ab}^c \equiv \oint \frac{dz}{2\pi i} z^{a+b} : \partial^a c(z) \partial^b b(z) :_c, \quad (3.23)$$

where the ordering is the conformal ordering of the bc CFT. The relation between the two types of W can be derived from (3.18) and is

$$\begin{aligned} W_{ab} &= \oint \frac{dz}{2\pi i} z^{a+b} : \partial^a c(z) \partial^b b(z) : = W_{ab}^c + \oint \frac{dz}{2\pi i} z^{a+b} \lim_{z' \rightarrow z} \partial^a \partial'^b \frac{z'^{1-\lambda} z^{\lambda-1} - 1}{z' - z} = \\ &= W_{ab}^c + S_{ab}. \end{aligned} \quad (3.24)$$

In particular the shifts used in the following are

$$\begin{aligned} S_{00} &= -(\lambda - 1), & S_{01} &= \frac{\lambda(\lambda - 1)}{2}, & S_{11} &= \frac{\lambda(\lambda - 1)(\lambda - 2)}{3} \\ S_{12} &= -\frac{\lambda(\lambda^2 - 1)(\lambda - 2)}{4}, & S_{22} &= -\frac{\lambda(\lambda^2 - 1)(\lambda - 2)(\lambda - 3)}{5}. \end{aligned} \quad (3.25)$$

The main step is to write down W_{ab}^c , and then W_{ab} , in terms of the modes of $X_L(z)$ and $X_R(\bar{z})$ as in the quadratic case.

Bosonization is carried out using the following relations between the fermionic and bosonic fields

$$\begin{aligned} : c(z) \partial b(z) :_c &= \frac{1}{2} : (\partial X_L(z))^2 :_c - \frac{i}{2} : \partial^2 X_L(z) :_c \\ : \partial c(z) \partial b(z) :_c &= -\frac{i}{3} : (\partial X_L)^3 :_c - \frac{i}{6} : \partial^3 X_L :_c \\ : \partial c(z) \partial^2 b(z) :_c &= \frac{1}{4} : (\partial X_L(z))^4 :_c - \frac{i}{2} : (\partial X_L(z))^2 \partial^2 X_L(z) :_c + \\ &\quad + \frac{1}{4} : (\partial^2 X_L(z))^2 :_c - \frac{i}{12} : \partial^4 X_L(z) :_c \\ : \partial^2 c(z) \partial^2 b(z) :_c &= -\frac{i}{5} : (\partial X_L)^5 :_c - i : \partial X_L(z) (\partial^2 X_L(z))^2 :_c - \frac{i}{30} : \partial^5 X_L(z) :_c \end{aligned} \quad (3.26)$$

(and the similar relations between \tilde{b}, \tilde{c} and X_R). From these and

$$L_0 = \sum n : c_{-n} b_n :_c, \quad (3.27)$$

the normal ordered integrals are easily calculated in terms of the bosonic modes, giving

$$\begin{aligned} W_{00} &= \alpha_0 - (\lambda - 1) \\ W_{01} &= -L_0 + \frac{\lambda(\lambda - 1)}{2} - \lambda \alpha_0 \end{aligned}$$

$$\begin{aligned}
W_{11} &= \frac{\alpha_0(1-\alpha_0^2)}{3} + \frac{\lambda(\lambda-1)(\lambda-2)}{3} - 2\alpha_0\mathcal{N} - A_3 \\
W_{12} &= \frac{\alpha_0(\alpha_0^2-1)(\alpha_0+2)}{4} - \frac{\lambda(\lambda^2-1)(\lambda-2)}{4} + \left(3\alpha_0(\alpha_0+1) - \frac{1}{2}\right)\mathcal{N} + \\
&\quad + \frac{1}{2}A_{2n^2} + \left(3\alpha_0 + \frac{3}{2}\right)A_3 + A_4 \\
W_{22} &= \frac{\alpha_0(\alpha_0^2-1)(\alpha_0^2-4)}{5} - \frac{\lambda(\lambda^2-1)(\lambda-2)(\lambda-3)}{5} + \\
&\quad + 2\alpha_0((2\alpha_0^2-3)\mathcal{N} + A_{2n^2}) - 3\left(A_3 - \frac{A_{3n^2}}{2}\right) + 6\alpha_0^2A_3 + 4\alpha_0A_4 + A_5,
\end{aligned} \tag{3.28}$$

with

$$\begin{aligned}
\mathcal{N} &= A_2 = \sum_{n>0} \alpha_{-n}\alpha_{+n} \\
A_{2n^2} &= \sum_{n>0} n^2 \alpha_{-n}\alpha_{+n} \\
A_3 &= \sum_{\substack{n_1, n_2, n_3 > 0 \\ n_1 + n_2 - n_3 = 0}} (\alpha_{-n_3}\alpha_{+n_1}\alpha_{+n_2} + \alpha_{-n_1}\alpha_{-n_2}\alpha_{+n_3}) \\
A_{3n^2} &= \sum_{\substack{n_1, n_2, n_3 > 0 \\ n_1 + n_2 - n_3 = 0}} \frac{n_1^2 + n_2^2 + n_3^2}{3} (\alpha_{-n_3}\alpha_{+n_1}\alpha_{+n_2} + \alpha_{-n_1}\alpha_{-n_2}\alpha_{+n_3}) \\
A_4 &= \sum_{\substack{n_1, n_2, n_3, n_4 > 0 \\ \sum \pm n_i = 0}} \left(\alpha_{-n_1}\alpha_{-n_2}\alpha_{-n_3}\alpha_{+n_4} + \right. \\
&\quad \left. + \frac{3}{2}\alpha_{-n_1}\alpha_{-n_2}\alpha_{+n_3}\alpha_{+n_4} + \alpha_{-n_1}\alpha_{+n_2}\alpha_{+n_3}\alpha_{+n_4} \right) \\
A_5 &= \sum_{\substack{n_1, \dots, n_5 > 0 \\ \sum \pm n_i = 0}} (\alpha_{-n_1}\alpha_{-n_2}\alpha_{-n_3}\alpha_{-n_4}\alpha_{+n_5} + \\
&\quad + 2\alpha_{-n_1}\alpha_{-n_2}\alpha_{-n_3}\alpha_{+n_4}\alpha_{+n_5} + 2\alpha_{-n_1}\alpha_{-n_2}\alpha_{+n_3}\alpha_{+n_4}\alpha_{+n_5} + \\
&\quad + \alpha_{-n_1}\alpha_{+n_2}\alpha_{+n_3}\alpha_{+n_4}\alpha_{+n_5}),
\end{aligned} \tag{3.29}$$

where by $\sum \pm n_i$ we mean, for each term, the sum of the indexes with their signs, for example for the first term in A_4 , $\sum \pm n_i = -n_1 - n_2 - n_3 + n_4$.

The last ingredients are the substitution for the Virasoro zero modes in terms of α_0 , \mathcal{N} and λ

$$L_0 = \frac{1}{2}\alpha_0^2 + \mathcal{N} - \left(\lambda - \frac{1}{2}\right)\alpha_0, \tag{3.30}$$

and the substitution for the constraint (3.22) that, introducing w as in [15], we can write as

$$\alpha_0 = \lambda - 1 + \frac{w}{2}, \quad \tilde{\alpha}_0 = \lambda - 1 - \frac{w}{2}. \tag{3.31}$$

The result for the $SU(N)$ hamiltonian must be independent of w (see [15]): we check that this is indeed the case.

The final form for the string hamiltonian in the generalized YM with quartic Casimir is

$$\begin{aligned}
H = \lambda_4 L & \left[\frac{\mathcal{N} + \tilde{\mathcal{N}}}{6} + \frac{7}{6N}(A_3 + \tilde{A}_3) + \right. \\
& + \frac{1}{N^2} \left(A_{2n^2} + \tilde{A}_{2n^2} + 2(A_4 + \tilde{A}_4) - \frac{13}{6}(\mathcal{N} - \tilde{\mathcal{N}})^2 \right) + \\
& + \frac{1}{N^3} \left(\frac{3}{2}(A_{3n^2} + \tilde{A}_{3n^2}) - 6(\mathcal{N} - \tilde{\mathcal{N}})(A_3 - \tilde{A}_3) + A_5 + \tilde{A}_5 \right) + \\
& \left. + \frac{6(\mathcal{N} - \tilde{\mathcal{N}})^2(A_3 + \tilde{A}_3)}{N^5} - \frac{3(\mathcal{N} - \tilde{\mathcal{N}})^4}{N^6} \right]. \tag{3.32}
\end{aligned}$$

As we can see, the final result is independent of w and λ . This is a non-trivial test of the correctness of the result, as the two parameters were present in the intermediate expressions in a very complex way.

To interpret this result first of all we recall that, as in the standard case, the powers of $g_s = \frac{1}{N}$ correspond to the factors $g_s^{-\chi}$ where χ is the Euler characteristic of the surface involved.

As in the standard quadratic Casimir case, we see that this hamiltonian presents a simple free propagation part, $\mathcal{N} + \tilde{\mathcal{N}}$ and an interaction vertex $V_3 = \frac{1}{N}(A_3 + \tilde{A}_3)$ that connects three strings (two string joining into one, or one splitting in two), conserving the total winding number and the orientation.

In this generalized theory there also new vertices $V_4 = \frac{1}{N^2}(A_4 + \tilde{A}_4)$ and $V_5 = \frac{1}{N^3}(A_5 + \tilde{A}_5)$ that simply connect four and five strings, still conserving the total winding number and the orientation. The power of N is the correct $g_s^{-\chi}$ factor.

The term $\frac{1}{N^2}(\mathcal{N} - \tilde{\mathcal{N}})^2$ is the same one has in the standard theory, and it is interpreted as usual as a contribution from microscopic tubes and handles connecting the sheets of each string or the sheets of two different strings. The contribution to the Euler characteristic for the addition of a tube or of an handle is -2 , so the $1/N^2$ factor is correct. In the same way $\frac{1}{N^6}(\mathcal{N} - \tilde{\mathcal{N}})^4$ can be interpreted as the contribution from a microscopic surface, topologically identified with a sphere with 4 holes, that connects 4 sheets. The -6 change of the Euler characteristic given by the connection of the sheets with this surface gives the factor $1/N^6$.

We have also other terms with a less clear geometric interpretation; for example we can interpret the terms $\frac{1}{N^3}(\mathcal{N} - \tilde{\mathcal{N}})(A_3 - \tilde{A}_3)$ and $\frac{1}{N^5}(\mathcal{N} - \tilde{\mathcal{N}})^2(A_3 + \tilde{A}_3)$ as combinations of a cubic vertex and handles. Although this interpretation is intricated and even not unique, this is not a limitation, since the important thing is that we have found an hamiltonian which resumes in a compact way all the perturbative aspects of the string theory.

4. Conclusions

In this paper we have investigated the two-dimensional generalized Yang Mills theory from

the point of view of the corresponding string description and in particular the calculation of the string hamiltonian.

We have focused only on the quartic Casimir case, but from this example we can extract some general features of the string description for the generic theory.

The string interpretation is similar to the standard case: the Hilbert space of the theory is the same (states of strings winding around the spatial direction) and the hamiltonian has a similar structure, it simply contains new kind of interaction vertices.

In particular for a Casimir of order k one obtains all the vertices connecting up to $(k + 1)$ strings; moreover in general there are many other interaction terms which are combinations of vertices and microscopic tubes or handles, for which we haven't found a general rule: it is not clear why for a specified Casimir there are particular terms and not others. But in general we can say that for a generic combination of higher Casimirs we expect to obtain a generic combination of the possible interaction terms.

Thus the string field theory hamiltonian summarizes quite compactly the perturbative features of the string description.

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A. The expression of the γ_k 's in terms of the l_i 's

In this appendix we present the calculation that we have done to write down the values of the gamma terms, and in particular the list (2.26). The gamma terms are

$$\gamma_k \equiv \frac{\chi_R(P_k)}{d_R} = k \frac{\chi_R(T_k)}{d_R}. \quad (\text{A.1})$$

Following Hamermesh [19] (Sect. 7-6, where h_i is our l_i) we obtain a formula for $\chi_R(T_k)/d_R$:

$$k \frac{\chi_R(T_k)}{d_R} = \frac{\sum_i l_i(l_i - 1) \cdots (l_i - (k - 1)) D[l_1, \dots, l_{i-1}, l_i - k, l_{i+1}, \dots, l_N]}{D[l_1, l_2 \dots l_N]}, \quad (\text{A.2})$$

where $D[l_1, l_2 \dots l_N] \equiv \prod_{i < j} (l_i - l_j)$

We reduce this expression to a polynomial of l_i using the following formula:

$$\begin{aligned} F_x[l_i^z] &\equiv \frac{\sum_i l_i^z D[l_1, \dots, l_{i-1}, l_i + x, l_{i+1}, \dots, l_N]}{D[l_1, l_2 \dots l_N]} = \sum_i l_i^z \prod_{j \neq i} \left(1 + \frac{x}{l_i - l_j}\right) = \\ &= \sum_{a=0}^z x^a \sum_{\substack{b_1, b_2 \dots b_{a+1} \in \{1 \dots N\} \\ b_1 < \dots < b_{a+1}}} E_{z, a+1}[l_{b_1}, \dots, l_{b_{a+1}}], \quad (\text{A.3}) \end{aligned}$$

where

$$E_{z,m}[x_1, x_2, \dots, x_m] = \sum_i \frac{x_i^z}{\prod_{j \neq i} (x_i - x_j)} = \sum_{\substack{a_1, a_2, \dots, a_m \geq 0 \\ \sum_j a_j = z - m + 1}} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}. \quad (\text{A.4})$$

We postpone the demonstration of these formulas at the end of the appendix. We use the formulas above to calculate $\gamma_1, \gamma_2, \gamma_3$ and γ_4 , that have the following simple expressions in terms of $F_x[l_i^\beta]$:

- γ_1 is $F_x[l_i]$ for $x = -1$
- γ_2 is $F_x[l_i(l_i - 1)] = F_x[l_i^2 - l_i] = F_x[l_i^2] - F_x[l_i]$ for $x = -2$.
- γ_3 is $F_x[l_i(l_i - 1)(l_i - 2)] = F_x[l_i^3 - 3l_i^2 + 2l_i] = F_x[l_i^3] - 3F_x[l_i^2] + 2F_x[l_i]$ for $x = -3$.
- γ_4 is $F_x[l_i(l_i - 1)(l_i - 2)(l_i - 3)] = F_x[l_i^4 - 6l_i^3 + 11l_i^2 - 6l_i] = F_x[l_i^4] - 6F_x[l_i^3] + 11F_x[l_i^2] - 6F_x[l_i]$ for $x = -4$.

We have (notice that $E_{a,a+1}[\dots] = 1$)

$$F_x[l_i] = \sum_i E_{1,1}[l_i] + x \sum_{i < j} E_{1,2}[l_i, l_j] = \sum_i l_i + x \binom{N}{2}, \quad (\text{A.5})$$

$$\begin{aligned} F_x[l_i^2] &= \sum_i E_{2,1}[l_i] + x \sum_{i < j} E_{2,2}[l_i, l_j] + x^2 \sum_{i < j < k} E_{2,3}[l_i, l_j, l_k] = \\ &= \sum_i l_i^2 + x(N-1) \sum_i l_i + x^2 \binom{N}{3}, \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} F_x[l_i^3] &= \sum_i E_{3,1}[l_i] + x \sum_{i < j} E_{3,2}[l_i, l_j] + x^2 \sum_{i < j < k} E_{3,3}[l_i, l_j, l_k] + \\ &+ x^3 \sum_{i < j < k < m} E_{3,4}[l_i, l_j, l_k, l_m] = \\ &= \sum_i l_i^3 + x((N-1) \sum_i l_i^2 + \sum_{i < j} l_i l_j) + x^2 \binom{N-1}{2} \sum_i l_i + x^3 \binom{N}{4}, \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} F_x[l_i^4] &= \sum_i E_{4,1}[l_i] + x \sum_{i < j} E_{4,2}[l_i, l_j] + x^2 \sum_{i < j < k} E_{4,3}[l_i, l_j, l_k] + \\ &+ x^3 \sum_{i < j < k < m} E_{4,4}[l_i, l_j, l_k, l_m] + x^4 \binom{N}{5} = \\ &= \sum_i l_i^4 + x \left[(N-1) \sum_i l_i^3 + x \sum_{i < j} (l_i^2 l_j + l_i l_j^2) \right] + x^2 \left[\binom{N-1}{2} \sum_i l_i^2 + \right. \\ &\left. + (N-2) \sum_{i < j} l_i l_j \right] + x^3 \binom{N-1}{3} \sum_i l_i + x^4 \binom{N}{5}. \end{aligned} \quad (\text{A.8})$$

After some calculation, we obtain the list

$$\begin{aligned}
\gamma_1 &= \sum_i l_i - \binom{N}{2} = \sum_i n_i = C_1 = n \\
\gamma_2 &= \sum_i l_i^2 - (2N-1) \sum_i l_i + \frac{N(N-1)(2N-1)}{3} \\
\gamma_3 &= \sum_i l_i^3 - 3N \sum_i l_i^2 - 3 \sum_{i<j} l_i l_j + \frac{9N^2 - 9N + 4}{2} \sum_i l_i + \\
&\quad - \frac{3}{8} N(N-1)(3N^2 - 3N + 2) \\
\gamma_4 &= \sum_i l_i^4 - 4 \sum_{i<j} (l_i^2 l_j + l_i l_j^2) - (4N+2) \sum_i l_i^3 + (16N-8) \sum_{i<j} l_i l_j + \\
&\quad + (8N^2 + 3) \sum_i l_i^2 - \frac{2}{3} (2N-1)(8N^2 - 8N + 9) \sum_i l_i + \\
&\quad + \frac{4}{7} N(N-1)(2N-1)(4N^2 - 4N + 7),
\end{aligned} \tag{A.9}$$

that we use in Sect. 2.2.

In the following we demonstrate the formulas (A.3), (A.4). We recall the definitions and the formulas to be proven:

Definition 1:

$$F_x[l_i^z] \equiv \frac{\sum_i l_i^z D[l_1, \dots, l_{i-1}, l_i + x, l_{i+1}, \dots, l_N]}{D[l_1, l_2, \dots, l_N]}. \tag{A.10}$$

Definition 2:

$$E_{z,m}[x_1, x_2, \dots, x_m] \equiv \sum_i \frac{x_i^z}{\prod_{j \neq i} (x_i - x_j)}. \tag{A.11}$$

Theorem 1:

$$\begin{aligned}
F_x[l_i^z] &= \sum_i l_i^z \prod_{j \neq i} \left(1 + \frac{x}{l_i - l_j} \right) = \\
&= \sum_{a=0}^z x^a \sum_{\substack{b_1, b_2, \dots, b_{a+1} \in \{1..N\} \\ b_1 < \dots < b_{a+1}}} E_{z, a+1}[l_{b_1}, \dots, l_{b_{a+1}}].
\end{aligned} \tag{A.12}$$

Theorem 2:

$$E_{z,m}[x_1, x_2, \dots, x_m] = \sum_{\substack{a_1, a_2, \dots, a_m \geq 0 \\ \sum_j a_j = z - m + 1}} x_1^{a_1} x_2^{a_2} \dots x_m^{a_m}. \tag{A.13}$$

The two theorems give the desired formulas.

Proof of Theorem 1:

From the Definition 1 of $F_x[.]$:

$$\begin{aligned}
F_x[l_i^z] &= \sum_i l_i^z \prod_{a, (a < i)} \left(1 - \frac{x}{l_a - l_i}\right) \prod_{b, (b > i)} \left(1 + \frac{x}{l_i - l_b}\right) = \\
&= \sum_i l_i^z \prod_{j, j \neq i} \left(1 + \frac{x}{l_i - l_j}\right) = \\
&= l_1^z \left(1 + \frac{x}{l_1 - l_2}\right) \left(1 + \frac{x}{l_1 - l_3}\right) \cdots \left(1 + \frac{x}{l_1 - l_N}\right) + \\
&\quad + l_2^z \left(1 + \frac{x}{l_2 - l_1}\right) \left(1 + \frac{x}{l_2 - l_3}\right) \cdots \left(1 + \frac{x}{l_2 - l_N}\right) + \cdots \\
&\quad \cdots + l_N^z \left(1 + \frac{x}{l_N - l_1}\right) \left(1 + \frac{x}{l_N - l_3}\right) \cdots \left(1 + \frac{x}{l_N - l_{N-1}}\right) =
\end{aligned} \tag{A.14}$$

then we can collect the powers of x easily:

$$\begin{aligned}
&= \sum_b l_b^z + x \sum_{b_1 < b_2} \sum_{i=1}^2 \frac{l_{b_i}^z}{\prod_{j=1..2, j \neq i} (l_{b_i} - l_{b_j})} + x^2 \sum_{b_1 < b_2 < b_3} \sum_{i=1}^3 \frac{l_{b_i}^z}{\prod_{j=1..3, j \neq i} (l_{b_i} - l_{b_j})} + \cdots \\
&\quad + x^a \sum_{b_1 < b_2 < b_3 < \dots < b_{a+1}} \sum_{i=1}^{a+1} \frac{l_{b_i}^z}{\prod_{j=1..(a+1), j \neq i} (l_{b_i} - l_{b_j})} + \cdots \\
&\quad + x^{N-1} \sum_{b_1 < b_2 < b_3 < \dots < b_N} \sum_{i=1}^N \frac{l_{b_i}^z}{\prod_{j=1..N, j \neq i} (l_{b_i} - l_{b_j})} =
\end{aligned} \tag{A.15}$$

in terms of $E_{z,m}$ previously defined:

$$= \sum_a^{N-1} x^a \sum_{b_1 < b_2 < \dots < b_{a+1} = 1..N} E_{z,a+1}[l_{b_1}, l_{b_2}, \dots, l_{b_{a+1}}], \tag{A.16}$$

and this is Theorem 1.

Proof of Theorem 2:

This identity is very simple for $m = 2$:

$$E_{z,2} = \frac{x_1^z}{x_1 - x_2} + \frac{x_2^z}{x_2 - x_1} = \frac{x_1^z - x_2^z}{x_1 - x_2} = x_1^{z-1} + x_2^{z-1} + x_1^{z-2} x_2 + x_1^{z-3} x_2^2 + \dots + x_1 x_2^{z-2}. \tag{A.17}$$

The proof is by induction. Defining

$$E'_{z,m} = \sum_{\substack{a_1, a_2, \dots, a_m \geq 0 \\ \sum_j a_j = z - m + 1}} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}, \tag{A.18}$$

the hypothesis of the induction is

$$E_{z,i} = E'_{z,i} \quad \forall i \in \{2, 3, \dots, m-1\} \quad \text{and} \quad \forall z \in \mathbb{N}, \tag{A.19}$$

and the thesis of the theorem is

$$E_{z,m} = E'_{z,m} \quad \forall z \in \mathbb{N}. \quad (\text{A.20})$$

Let us start from $E'_{z,m}$ and try to write it in terms of $E_{z,m}$ using the hypothesis of the induction for $m-1$, isolating one of the x (it does not matter which x , we isolate x_m and use the hypothesis on x_1, x_2, \dots, x_{m-1} , but they are all equivalent):

$$E'_{z,m} = \sum_{\substack{a_1, a_2, \dots, a_{m-1}, a_m \geq 0 \\ \sum_{j=1}^{m-1} a_j = (z-1) - a_m - (m-1) + 1}} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}. \quad (\text{A.21})$$

Using the hyp.:

$$\begin{aligned} E'_{z,m} &= \sum_{a_m \geq 0} \sum_{i=1}^{m-1} \frac{x_i^{z-1-a_m}}{\prod_{j=1, \dots, m-1; j \neq i} (x_i - x_j)} x_m^{a_m} = \\ &= \sum_{i=1}^{m-1} \left[\frac{x_i^z}{\prod_{j=1, \dots, m; j \neq i} (x_i - x_j)} + \frac{x_m^z}{\prod_{j=1, \dots, m-1; j \neq i} (x_i - x_j)(x_m - x_j)} \right], \end{aligned} \quad (\text{A.22})$$

where we have used the formula for $m=2$ that is simply:

$$\sum_{a_m \geq 0} x_i^{z-1-a_m} x_m^{a_m} = \frac{x_i^z}{x_i - x_m} + \frac{x_m^z}{x_m - x_i}. \quad (\text{A.23})$$

Now we have:

$$\begin{aligned} E'_{z,m} &= \sum_{i=1}^m \frac{x_i^z}{\prod_{j \neq i}^m (x_i - x_j)} + \\ &+ x_m^z \left[\sum_{i=1}^{m-1} \frac{1}{\prod_{j \neq i}^{m-1} (x_i - x_j)(x_m - x_i)} - \frac{1}{\prod_{j \neq m}^m (x_m - x_j)} \right] = \\ &= E_{z,m} - x_m^z \left[\sum_{i=1}^{m-1} \frac{1}{\prod_{j \neq i}^m (x_i - x_j)} + \frac{1}{\prod_{j \neq m}^m (x_m - x_j)} \right] = \\ &= E_{z,m} - x_m^z \sum_{i=1}^m \frac{1}{\prod_{j \neq i}^m (x_i - x_j)} = E_{z,m} - x_m^z E_{0,m}. \end{aligned} \quad (\text{A.24})$$

Thus we have

$$E'_{z,m} = E_{z,m} - x_m^z E_{0,m}. \quad (\text{A.25})$$

Notice that we know that $E_{0,i} = 0$ only for i up to $m-1$ from the induction hypothesis. But now we use the fact that we can isolate a generic x : the actual formula is

$$E'_{z,m} = E_{z,m} - x_j^z E_{0,m} \quad \forall j \in \{1, 2, \dots, m\}, \quad (\text{A.26})$$

thus if we consider this for $j=a$ and $j=b$ with $a \neq b$ and take the difference we have:

$$(x_a^z - x_b^z) E_{0,m} = 0 \quad \text{valid for generic } x \quad \Rightarrow E_{0,m} = 0. \quad (\text{A.27})$$

Finally we have the thesis

$$E'_{z,m} = E_{z,m}. \quad (\text{A.28})$$

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